



Upper bounds for ropelength as a function of crossing number

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Received 15 October 2002; received in revised form 12 March 2003

Abstract

This paper provides bounds for the ropelength of a link in terms of the crossing numbers of its prime components. As in earlier papers, the bounds grow with the square of the crossing number; however, the constant involved is a substantial improvement on previous results. The proof depends essentially on writing links in terms of their arc-presentations, and has as a key ingredient Bae and Park's theorem that an n -crossing link has an arc-presentation with less than or equal to $n + 2$ arcs. © 2003 Elsevier B.V. All rights reserved.

Keywords: Ropelength; Crossing number; Arc-presentations; Geometric knot theory

1. Introduction

The *ropelength* of a space curve is defined to be the quotient of its length by its *thickness*, where thickness is the radius of the largest embedded tubular neighborhood around the curve. For a knot or link type L , we define the ropelength $\text{Rop}(L)$ to be the minimum ropelength of all curves with the given link type. This minimum ropelength is a link invariant which measures the topological complexity of the link, much like crossing number, or bridge number, in classical knot theory.

It has been shown that every link type contains at least one $C^{1,1}$ *tight* representative which achieves this minimum ropelength [3,8]. Much effort has been invested in the project of finding lower bounds for the ropelength of various link types in terms of classical topological invariants, such as the crossing number [2,3,11].

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In this paper, we are interested in a converse problem: given a link type L of crossing number $c(L)$, can we guarantee the existence of a representative curve with ropelength less than some function of $c(L)$? That is, can we find *upper* bounds on ropelength in terms of crossing number? Our main theorem states the following:

Theorem 1. *If L is a non-split link, then*

$$\text{Rop}(L) \leq 1.64 c(L)^2 + 7.69 c(L) + 6.74. \quad (1)$$

In particular, this bound holds for prime links.

Our Theorem 2 gives similar bounds for composite links.

Other groups [3,10] have attacked this problem by finding upper bounds on the number of edges required to embed a given link L in the unit lattice (the *lattice number* $k(L)$ of the link), and then observing that $\text{Rop}(L) < 2k(L)$ [5]. Both proofs rely on laying out a diagram of the knot as a graph in a planar grid and then adding bridges to form overcrossings. In this context, it has been observed that constructing a particular diagram of a link with crossing number $c(L)$ may require ropelength $O(c(L)^2)$ [10]. These authors have obtained the weaker bounds $\text{Rop}(L) < 24 c(L)^2$ [3], and $\text{Rop}(L) < 25 c(L)^2$ [10]. Johnston's algorithm, like ours, produces an explicit realization of the knot in space, while the approach of [3] is less constructive. By contrast, our methods are more three-dimensional and are not based on grid or lattice embeddings. Instead of using a planar diagram of a knot, we base our construction on Peter Cromwell's idea of *arc-presentations* [4].

Diao, Ernst, and Yu have recently used an improved lattice embedding method to establish $O(c(L)^{3/2})$ upper bounds for ropelength. They obtain the inequality

$$\text{Rop}(L) \leq 34 c(L)^{3/2} + 42 c(L) + 22 c(L)^{1/2} + 22, \quad (2)$$

for links which have a minimal crossing number diagram containing a Hamiltonian cycle (which they call *minimally Hamiltonian*), and the general inequality

$$\text{Rop}(L) \leq 272 c(L)^{3/2} + 168 c(L) + 44 c(L)^{1/2} + 22 \quad (3)$$

for all (non-split) links [6, Theorem 5.4]. Comparing these bounds with those of Theorem 1, we see that our bound is preferable for knots with crossing number less than 470 in the special case, and 27,700 in general.

2. The definition of ropelength

The ropelength of a curve is defined to be the quotient of length by the radius of the largest embedded tubular neighborhood around the curve. This radius is called the *thickness* of the curve. For C^2 curves, this radius is locally controlled by curvature and globally controlled by distances of self-approach between various regions of the curve. Formally, we write

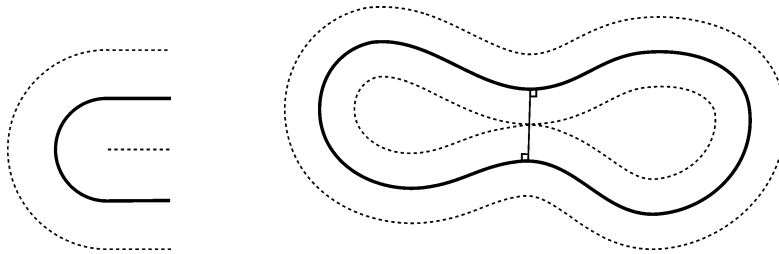


Fig. 1. These are two curves of unit thickness in the plane with their largest embedded tubular neighborhoods. In the left curve, thickness is controlled by curvature while in the right curve, thickness is controlled by the length of the doubly-critical chord shown.

Definition 1. The *thickness* of a C^2 curve c is given by

$$\tau[c] := \min \left\{ \min_s \frac{1}{\kappa(s)}, \frac{\text{dcsd}(c)}{2} \right\}, \quad (4)$$

where $\kappa(s)$ is the curvature of c at s , and $\text{dcsd}(c)$ is the shortest *doubly-critical self-distance* of c ; that is, the length of the shortest chord of c which is perpendicular to the tangent vector c' at both endpoints.

We can extend this definition to $C^{1,1}$ curves by adjusting our idea of the radius of curvature as follows (cf. [3]):

Definition 2. Let s be a point on a $C^{1,1}$ curve. Consider a decreasing sequence of open neighborhoods U_n of s . The *infimal radius of curvature* at s is given by

$$\inf_{U_n} \left\{ \inf_{t \in U_n} \frac{1}{\kappa(t)} \right\}, \quad (5)$$

where the inner infimum is restricted to t in U_n such that $\kappa(t)$ exists.

Fig. 1 shows examples of curves where thickness is controlled by curvature and by the doubly-critical self-distance.

Gonzalez and Maddocks have given another definition of thickness which looks somewhat less natural, but is often more useful. (See [9] for details.) Another useful way to look at thickness comes from Federer's notion of *reach*, which agrees with the thickness for curves [7].

Definition 3. The *reach* of a set S inside \mathbb{R}^n is the greatest non-negative r so that each point within distance r of S has a unique nearest neighbor in S .

3. Arc-presentations

We start with a definition:

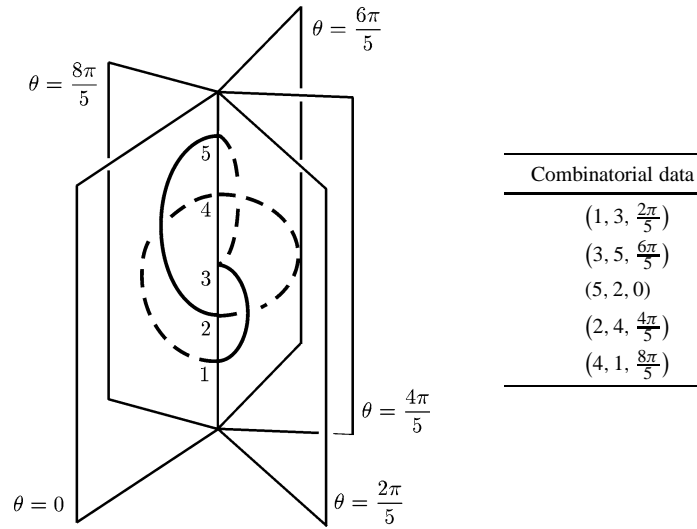


Fig. 2. This figure shows an arc-presentation for a trefoil knot. The presentation has arc-index 5. To the right we see the combinatorial data which describes this arc-presentation: 5 triples in the form (x_i, y_i, θ_i) , each indicating an arc from level x_i to level y_i on page θ_i of the “5-page book” shown on the left.

Definition 4. An *arc-presentation* of a link L is an embedding of L in a finite collection of α open half-planes arrayed around a common axis, or binding, so that the intersection of L with each half-plane is a single simple arc. The number of half-planes α is called the *arc-index* of the arc-presentation. The minimal arc-index over all arc-presentations of a link L is an invariant of the link type.

By isotopy, we can arrange that L intersects the axis only at the points $1, \dots, \alpha$. We call these the *levels* of the arc-presentation. Such an arc-presentation is then specified by combinatorial data: a collection of α triples in the form (x_i, y_i, θ_i) , where each denotes an arc from level x_i to level y_i on the half-plane at angle θ_i around the axis.

Fig. 2 shows an arc-presentation for the trefoil and the corresponding set of triples.

We will assemble our ropelength bounds from two ingredients. First, we define a notion of the total distance travelled by the arcs in an arc-presentation:

Definition 5. The *total skip* of an arc-presentation A , denoted $\text{Skip}(A)$, is

$$\text{Skip}(A) = \sum_{i=1}^{\alpha} |x_i - y_i|. \quad (6)$$

For a given arc-presentation we can construct a realization of the knot in space with ropelength bounded in terms of $\text{Skip}(A)$ and α :

Proposition 1. *An arc-presentation A with arc-index α can be realized with ropelength smaller than*

$$\frac{2\alpha}{\tan(\pi/\alpha)} + (\pi - 2)\alpha + 2\text{Skip}(A). \quad (7)$$

For the arc-presentation of the trefoil in Fig. 2, we have $\alpha = 5$ and $\text{Skip}(A) = 12$; so Proposition 1 yields an upper bound on the ropelength of the trefoil of about 43.47. Numerical experiments estimate the ropelength of the tight trefoil to be about 32.66 [12], so the slack in our estimate is about 33% of the total value. Fig. 3 shows the tubular neighborhoods of this trefoil knot and an arc-presentation of the knot 7_1 as realized by the algorithm in the proof of Proposition 1.

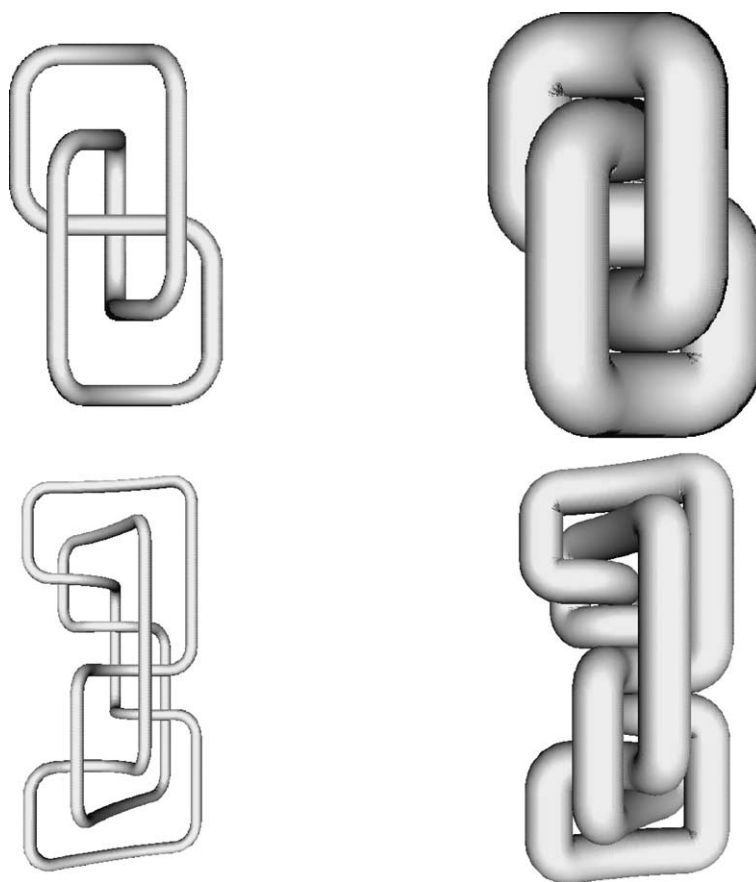


Fig. 3. Here we see a trefoil knot (top left) and a 7_1 knot (bottom left) together with the tubular neighborhoods around them constructed by the proof of Proposition 1. Our trefoil knot appears much tighter: its ropelength (43.47) is proportionally closer to the minimum ropelength for its knot type (32.66) than the ropelength of our 7_1 knot (97.05) is to the minimum for its knot type (61.40 [12]).

Further, if we can bound $\text{Skip}(A)$ in general, we will be able to draw conclusions about the ropelength of an arbitrary link. A combinatorial argument yields:

Proposition 2. *If an arc-presentation A has arc-index α , then*

$$\text{Skip}(A) \leq \begin{cases} \frac{\alpha^2-1}{2} & \text{if } \alpha \text{ is odd,} \\ \frac{\alpha^2}{2} & \text{if } \alpha \text{ is even.} \end{cases} \quad (8)$$

This bound is sharp.

It is shown in [1] that any non-split link L admits an arc-presentation with $\alpha \leq c(L) + 2$. This result, when coupled with the previous two propositions, gives Theorem 1. We obtain an even stronger statement for composite links:

Theorem 2. *If L is a non-split composite link with prime components L_1, L_2, \dots, L_n , then*

$$\text{Rop}(L) \leq 1.64 \sum_{i=1}^n c(L_i)^2 + 7.69 \sum_{i=1}^n c(L_i) + 6.74n. \quad (9)$$

4. Proofs of the key propositions and theorems

Proof of Proposition 1. We would like to take an arc-presentation A for L as a template for constructing an embedding of L with unit thickness. We will then bound the length of this embedding in terms of the arc-index and the total skip of A .

We begin by constructing a right regular polygonal prism $P \times [0, 2\alpha]$, where P is a regular polygon with α sides of length 2. This prism will serve as the binding of A ; each vertical face of the prism will correspond to an open half-plane in the arc-presentation A . We divide the prism vertically into α floors, each a prism of height 2, which will represent the α levels of the arc-presentation A .

We can now construct a link isotopic to L . First, represent the arcs of A by α handles outside the prism which join different floors on the same vertical face. We will refer to these handles as *fins*. Next, add α circular sections inside the prism which join different vertical faces on the same floor. These sections represent the junctions between arcs on the binding of the open book described by A .

We must show that this construction can be accomplished with a unit thickness curve and then compute the length of that curve.

4.1. The fins

Let us denote the fins $\mathcal{F}_1, \dots, \mathcal{F}_\alpha$. Each fin consists of two quarter-circles of unit radius, joined by a straight vertical segment. Each fin joins two points on a vertical face of the prism and is contained in a $2\alpha \times 2 \times 2$ rectangular box extending radially from a vertical face of the prism.

Since the \mathcal{F}_i 's stay outside the prism and each is contained in a different box, the tubes around the fins are pairwise disjoint, and disjoint from the tubes surrounding regions of

the curve inside the prism. Given that each fin has curvature bounded above by 1 and no doubly-critical chords, this means that the fins can be constructed with a unit-thickness curve.

Claim 1. *If $\text{Rop}(\mathcal{F}_i)$ denotes the length of the segment of the curve \mathcal{F}_i , then*

$$\sum_{i=1}^{\alpha} \text{Rop}(\mathcal{F}_i) = (\pi - 2)\alpha + 2 \text{Skip}(A). \quad (10)$$

Proof. Suppose that \mathcal{F}_i travels from floor x_i to floor y_i of the prism. The total vertical distance covered by the fin is $2|x_i - y_i|$ (recall that each floor has height 2). However, the quarter-circles on each end of the fin cover a vertical distance of 2 units. Thus, the straight segment has length $2|x_i - y_i| - 2$, and the total length of the fin is $\pi - 2 + 2|x_i - y_i|$. Summing over $i = 1, \dots, \alpha$ and using Definition 5 proves the claim. \square

4.2. The binding prism

We denote the sections of the curve inside each floor of the binding prism by $\mathcal{B}_1, \dots, \mathcal{B}_{\alpha}$. Each \mathcal{B}_i is a circular arc joining the midpoints of two edges of the regular polygon which is the cross-section of the prism as shown in Fig. 4.

Because the sides of the polygon have length 2, each of these is an arc of a circle of radius at least one; so each arc has curvature bounded above by one. Further, since each floor has height 2 and only one \mathcal{B}_i lies in each floor, the tubes around each of the \mathcal{B}_i are disjoint. Thus these \mathcal{B}_i can be constructed with a tube of unit thickness.

Claim 2. *If $\text{Rop}(\mathcal{B}_i)$ denotes the length of the segment of the curve \mathcal{B}_i , then*

$$\sum_{i=1}^{\alpha} \text{Rop}(\mathcal{B}_i) \leq \frac{2\alpha}{\tan(\pi/\alpha)}. \quad (11)$$

Proof. Each of these circular arcs is contained in a sector of the circle inscribed within the polygonal cross-section of the prism as shown in Fig. 5. Since each arc is convex, its length is bounded above by the diameter of the inscribed circle. This diameter is exactly $2 \cot(\pi/\alpha)$. Summing over $i = 1, \dots, \alpha$ proves the claim. \square

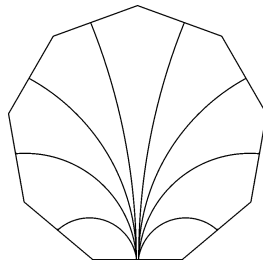


Fig. 4. The sections of our curve \mathcal{B}_i within the binding prism are circular arcs joining the midpoints of edges of the cross-section of the prism. The plane of this picture is located in the center of a floor of the prism.

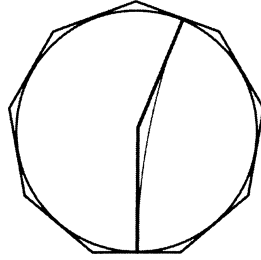


Fig. 5. Each of the paths \mathcal{B}_i through the floors of the binding prism is a circular arc connecting two sides of the polygon which is that prism's cross-section. Here we see that each of these arcs is contained within a sector of the circle inscribed within that polygon. Since each arc is a convex curve, this means that its length is bounded by the length of the two radii which bound the sector. That is, it is bounded by the diameter of the inscribed circle.

Combining Claims 1 and 2 yields the statement of Proposition 1. \square

Proof of Proposition 2. Our job is to find an upper bound for $\text{Skip}(A) = \sum_{i=1}^{\alpha} |x_i - y_i|$. We first observe that the difference $|x_i - y_i|$ is one unit larger than the number of levels skipped over. For example, jumping from level 3 to level 6, a difference of 3 levels, skips the fourth and fifth levels. Thus, we can rewrite the sum

$$\text{Skip}(A) = \alpha + \sum_{i=1}^{\alpha} \{\text{number of levels skipped by the arc } (x_i, y_i, \theta_i)\}. \quad (12)$$

Notice that any level j contributes to the above sum exactly when it is skipped over. We can rewrite our sum in terms of j as

$$\begin{aligned} \text{Skip}(A) &= \alpha + \sum_{j=1}^{\alpha} \{\text{number of times level } j \text{ is skipped}\} \\ &= \alpha + \sum_{j=1}^{\lfloor \alpha/2 \rfloor} \{\text{number of times level } j \text{ is skipped}\} \\ &\quad + \sum_{j=0}^{\alpha - \lfloor \alpha/2 \rfloor - 1} \{\text{number of times level } \alpha - j \text{ is skipped}\}, \end{aligned} \quad (13)$$

where in the final equality we have split the second half of the sum off and let $j \mapsto \alpha - j$.

Now we bound the number of times level j is skipped over. The only way to hop over j from a higher level is to land on a lower level. There are $j - 1$ levels below the j th on which such a jump can land. Further, each of these levels can act as a launch pad for a jump back up which crosses the j th level again. This gives at most $2(j - 1)$ skips over level j . Similarly, the number of times we can skip over the $(\alpha - j)$ th level is twice the number of levels above it, or $2j$.

For even α , these estimates are sharp (as we will see below). However, when level $\alpha - j$ is the central level of an arc-presentation with $2k + 1$ levels ($j = k = (\alpha - 1)/2$), the situation is slightly different. Here all of the j levels above the middle cannot be initial

and terminal levels of arcs which skip level $\alpha - j$. For if so, then no arcs land on level $\alpha - j$, and we could have eliminated level $\alpha - j$ from the original arc-presentation. Thus level $\alpha - j$ is skipped at most $2j - 1 = \alpha - 2$ times.

Inserting these bounds into Eq. (13), we apply the sum formulae for arithmetic progressions. When α is odd, we get

$$\text{Skip}(A) \leq \alpha + \sum_{j=1}^{(\alpha-1)/2} 2(j-1) + \sum_{j=0}^{(\alpha-3)/2} 2j + (\alpha-2) = \frac{\alpha^2 - 1}{2}. \quad (14)$$

If α is even, the proof is similar.

We now construct arc-presentations which show that these results are sharp. Consider the arc-presentation with even arc-index $\alpha = 2k$ described by the data

$$(\alpha, \alpha/2, \theta_1), (\alpha/2, \alpha-1, \theta_2), (\alpha-1, \alpha/2-1, \theta_3), (\alpha/2-1, \alpha-2, \theta_4), \\ \dots, (\alpha/2+1, 1, \theta_{2k-1}), (1, \alpha, \theta_{2k}).$$

If we add up the lengths of the jumps, we get

$$\text{Skip}(A) = \alpha^2/2. \quad (15)$$

The same approach yields a realization of A so that $\text{Skip}(A) = (\alpha^2 - 1)/2$ for odd α . \square

Proof of Theorem 1. Taylor's theorem gives the approximation $1/\tan(x) \leq 1/x - x/3$ for $x > 0$. Via Propositions 1 and 2 we gather that

$$\begin{aligned} \text{Rop}(L) &\leq \frac{2\alpha}{\tan(\pi/\alpha)} + (\pi-2)\alpha + \alpha^2 \\ &\leq (2/\pi + 1)\alpha^2 + (\pi-2)\alpha - 2\pi/3. \end{aligned} \quad (16)$$

By Bae and Park [1], for any non-split link L there exists an arc-presentation with $\alpha \leq c(L) + 2$. Inserting this into the above bound for ropelength yields

$$\text{Rop}(L) \leq (2/\pi + 1)c(L)^2 + (8/\pi + 2 + \pi)c(L) + (8/\pi + 4\pi/3), \quad (17)$$

and each of these constants evaluates to something smaller than the approximations given in the statement of the theorem. To gain the final remark in the theorem, we note that any prime link L is non-split (otherwise it would consist of split components L_1 and L_2 and would admit the non-trivial factors L_1 and L_2 union a split unknot). \square

Proof of Theorem 2. The strategy for this proof is to arrange the prime components of our composite link so that we can make use of the bounds given by Theorem 1. So suppose that we have found arc-presentations with minimal arc-index for these components and embedded them as unit-thickness curves L_1, \dots, L_n according to the algorithm of Proposition 1.

We will now prove that for any prime links L_1 and L_2 , we can construct an embedding of the curve $L_1 \# L_2$ with ropelength less than the sum of the individual bounds given by Theorem 1. This is all that is required to complete the proof of our theorem since the bound in the statement is just the sum of the bounds obtained for the L_i by Theorem 1.

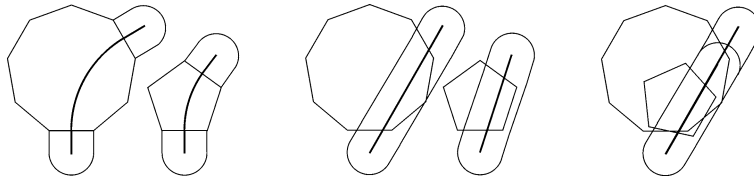


Fig. 6. We look down on a knot of arc-index 5, whose binding prism is shown by the small pentagon at right, preparing to be joined to a knot of arc-index 9, whose binding prism is shown by the large nonagon at left. The leftmost pair of figures shows the original position of the top and bottom arcs of these knots, while the middle pair of figures shows these arcs “straightened” to prepare for the connect sum. The rightmost pair of figures shows the two binding prisms in the correct relative position for the connect sum.

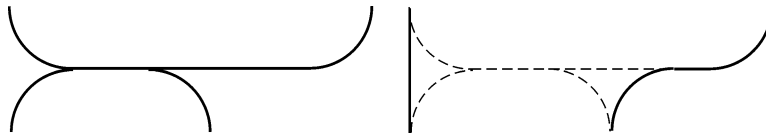


Fig. 7. This figure shows the extreme arcs of the two components of the connect sum, straightened, and aligned with one another on the left. On the right, we see the new curve. Two quarter-circles on the left have been replaced with a straight line segment; the lower quarter-circle has been rotated to face right; the lower horizontal segment (of length ℓ_1) has been deleted; and the upper horizontal segment (of length ℓ_2) has been replaced by a horizontal segment of length $\ell_2 - \ell_1 - 2$. Since these changes all reduce length, the curve on the right is strictly shorter.

We begin by preparing L_1 and L_2 . The top floor of L_1 contains only a single horizontal circular arc joining the centers of two sides of the binding prism. Since no fins jump over this level, we may rotate these quarter-circles to face one another and replace the horizontal circular arc with a horizontal line segment of shorter length without changing thickness or knot type. We do the same for the bottom floor of L_2 . This procedure is shown in Fig. 6.

We now arrange L_1 and L_2 in space so that the horizontal segments are collinear and share an endpoint. If we keep each oriented so that its floors are horizontal, the only overlap between the tubes surrounding each curve occurs on the shared floor. At the shared endpoint, we may delete two quarter-circles and replace them with a vertical line segment of length 2. We could keep track of this savings and get a slightly better constant term in the statement of Theorem 2. For each prime component we add, we save $\pi - 2$ in length.

Handling the other endpoints of the curve will prove to be a little more work. We may assume that both line segments lie along the x -axis with the shared endpoint at the origin. Suppose L_2 's segment has length ℓ_2 , while L_1 's segment has the smaller length ℓ_1 .

We now rotate the remaining vertical quarter-circle of L_1 to face the corresponding quarter-circle of L_2 . If $\ell_1 \leq \ell_2 - 2$, we may replace both horizontal line segments with a single, shorter horizontal line segment joining the ends of these vertical quarter circles to obtain the desired curve. See Fig. 7.

If $\ell_1 > \ell_2 - 2$, we cannot simply connect the endpoints of the quarter-circles after rotating the lower quarter-circle to face right. The resulting curve would have cusps on both ends. We solve this problem by finding a line tangent to both circles and following the composite path shown in Fig. 8.

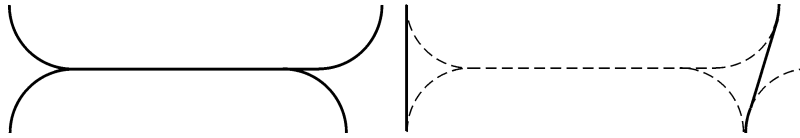


Fig. 8. This figure shows the two extreme arcs of the components in the case where $\ell_1 > \ell_2 - 2$. When we rotate the lower quarter-circle to face right, it cannot be joined by a horizontal straight segment to the upper quarter-circle to create a $C^{1,1}$ curve; instead we find the diagonal line tangent to both quarter-circles and follow the composite path shown.

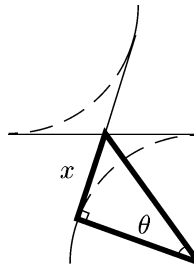


Fig. 9. This detailed figure enlarges the right-hand side of Fig. 8. Consider the triangle with the following vertices: the point of tangency of the diagonal segment with the lower circle, the center of the lower circle, and the midpoint of the diagonal segment. The portion of the lower quarter-circle replaced by this half of the line segment has length 2θ (again by symmetry). The length of this portion of the line segment is given by x .

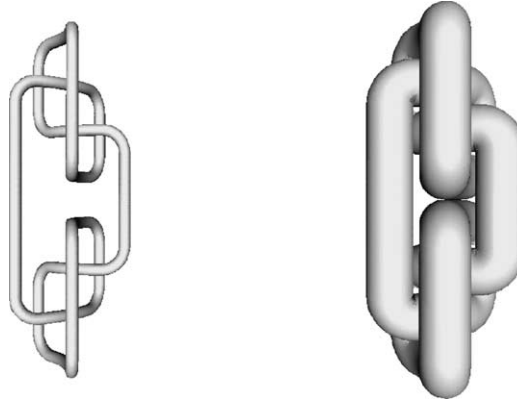


Fig. 10. Here we see the results of the construction of Theorem 2. Two mirror-image trefoil knots, generated by the method of Proposition 1 from the arc-presentation given in Fig. 2, have been joined by the methods of Theorem 2 to obtain the composite knot $3_1 \# \overline{3_1}$.

It is less obvious that these changes reduce length. To see that they do, we consider the diagonal line tangent to both circles shown in Fig. 8. Since both circles are also tangent to a horizontal line, by symmetry this horizontal line cuts the diagonal line in half. Consider Fig. 9. We need only show that half of the diagonal line (labelled x in the figure) is shorter than the portion of the quarter-circle it replaces (twice the angle θ).

Since the lower quarter-circle has unit radius, this amounts to proving that $\tan \theta \leq 2\theta$ for $0 \leq \theta \leq \pi/4$. This is shown by a simple computation.

Since the resulting curve remains $C^{1,1}$, is still of unit thickness, and has less length than the total length of the initial curves, this completes the proof. \square

An example of this construction is shown in Fig. 10.

Acknowledgements

We thank many of our colleagues for helpful conversations on these topics over several years, including Yuanan Diao, Heather Johnston, Rob Kusner and John Sullivan. We are indebted to the VIGRE program at the University of Georgia, and to the other members of our VIGRE research group: Ted Ashton, Kenny Little, Heunggi Park, Darren Wolford, and Nancy Wrinkle. In addition, Cantarella would like to acknowledge the generous support of the National Science Foundation Postdoctoral Research Fellowship Program under grant number DMS-99-02397 and the Geometric Analysis Program under grant DMS-02-04682, and Faber and Mullikin would like to recognize the gracious support of the VIGRE fellowship program (DMS 0089927) at UGA. Finally, Mullikin would like to thank Melinda Smith for her editing and support. Faber extends many thanks to Alana Cole for her patience and encouragement of his enduring enthusiasm.

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